NAMIBIA UNIVERSITY
OF SCIENCE AND TECHNOLOGY

FACULTY OF HEALTH, APPLIED SCIENCES AND NATURAL RESOURCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUALIFICATION: Bachelor of science in Applied Mathematics and Statistics	
QUALIFICATION CODE: 07BSAM	**LEVEL:** 6
COURSE CODE: SIN601S	**COURSE NAME:** STATISTICAL INFERENCE 2
SESSION: JANUARY 2023	**PAPER:** THEORY
DURATION: 3 HOURS	**MARKS:** 100

SUPPLEMENTARY / SECOND OPPORTUNITY EXAMINATION QUESTION PAPER	
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MODERATOR:	Dr D. NTIRAMPEBA

INSTRUCTIONS

1. There are 5 questions, answer ALL the questions by showing all the necessary steps.
2. Write clearly and neatly.
3. Number the answers clearly.
4. Round your answers to at least four decimal places, if applicable.

PERMISSIBLE MATERIALS

1. Nonprogrammable scientific calculator

THIS QUESTION PAPER CONSISTS OF 3 PAGES (Including this front page)

Question 1 [28 Marks]

1.1. Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics of n independently and identically distributed continuous random variables $X_1, X_2, \ldots, X_n$ with probability density function f and cumulative distribution function F. Then, the cumulative distribution function of r^{th} order statistics, $F_{Y_r}(y)$ is given by

$$F_{Y_r}(y) = \sum_{k=r}^{n} \binom{n}{k} (F_X(y))^k (1 - F_X(y))^{n-k}$$

Use this result to show that the cumulative distribution of the minimum statistic is given by

$$F_{Y_1}(y) = 1 - (1 - F(y))^n.$$ [4]

1.2. Let $Y_1 < Y_2 < \cdots < Y_5$ be the order statistics of 5 independently and identically distributed continuous random variables $X_1, X_2, \ldots, X_5$ with pdf f given by

$$f_X(x) = \begin{cases} 6x^2 & for\ 0 < x < 1 \\ 0 & otherwise \end{cases}$$

Then

1.2.1. Show that the cumulative density function of X is, $F_X(x) = 2x^3$ [2]

1.2.2. find the pdf of the r^{th} order statistics [3]

1.2.3. find the pdf of the minimum order statistics [3]

1.2.4. find the pdf of the maximum order statistics [3]

1.2.5. find the pdf of the median [4]

1.2.6. find the joint pdf of the 1st and 5th order statistics [5]

Hint: $f_{Y_i,Y_j}(y_i, y_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}[F(y_i)]^{i-1} f(y_i)[F(y_j) - F(y_i)]^{j-i-1} f(y_j)[1 - F(y_j)]^{n-j}$

$f_{Y_r}(y) = \frac{n!}{(r-1)!(n-r)!} f_X(y)[F_X(y)]^{r-1}[1 - F_X(y)]^{n-r}$

1.3. Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics of n independently and identically distributed continuous random variables $X_1, X_2, \ldots, X_n$ with standard normal, $N(0, 1)$, then find the joint pdf $Y_1, Y_2, \ldots, Y_n$. [4]

Question 2 [11 Marks]

2.1. Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed random variable with normal distribution having $E(X_i) = \mu$ and $V(X_i) = \sigma^2$. Then show, using the moment generating function, that $Y = \sum_{i=1}^{n} X_i$ has a normal distribution with mean $\mu_Y = n\mu$ and variance $\sigma_Y^2 = n\sigma^2$. (**Hint:** If $X \sim N(\mu, \sigma^2)$, then $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$). [8]

2.2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Then find the variance of $S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$. Hint: $(n-1)\frac{S^2}{\sigma^2} \sim \chi^2(n-1)$ with mean $(n-1)$ and variance $2(n-1)$ [3]

Question 3 [30 Marks]

3.1. The length of life of a component operating in guidance control system for missiles is assumed to follow a Weibull distribution with density function

$$f(x_i|\lambda) = \begin{cases} \frac{k}{\lambda}\left(\frac{x_i}{\lambda}\right)^{k-1} e^{-\left(\frac{x_i}{\lambda}\right)^k}, x_i \geq 0 \\ 0, \quad elsewhere. \end{cases}$$

If the parameter k is assumed to known, then find the MLE of λ. [10]

3.2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal population with mean μ and variance σ^2. That is

$$f(x_i|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2} \ for \ -\infty < x < \infty; -\infty < \mu < \infty \text{ and } \sigma^2 > 0$$

3.2.1. What are the method of moment estimators of the mean μ and variance σ^2? [9]

3.2.2. If μ is known, then show that $\sum_{i=1}^{n}(x_i - \mu)^2$ is sufficient statistic for σ^2. [5]

3.2.3. If σ^2 assumed to be known, derive the $100(1-\alpha)\%$ CI for μ using the pivotal quantity method. [6]

Question 4 [21 Marks]

4. Let $X_1, X_2, \ldots, X_n$ be and independent Bernoulli random variables with probability of success p and probability mass function

$$f(x_i|p) = \begin{cases} p^{x_i}(1-p)^{1-x_i} & for \ x_i = 0, 1 \\ 0 & otherwise \end{cases}$$

4.1. Using the mgf of X, show that the mean and variance of X_i are p and $p(1-p)$, respectively. (Hint: $M_X(t) = pe^t + (1-p)$). [5]

4.2. Show that the $\bar{X}$ is a minimum variance unbiased estimator (MVUE) of p. [16]

Question 5 [10 Marks]

5. Suppose the prior distribution of θ is uniform over the interval (2, 5) with pdf given by

$$h(\theta) = \begin{cases} \frac{1}{3} & if \ 2 < \theta < 5 \\ 0 & otherwise \end{cases}$$

Given θ, X is uniform over the interval $(0, \theta)$ with pdf given by

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} & if \ 0 < x < \theta \\ 0 & otherwise \end{cases}$$

What is the Bayes' estimate of θ for an **absolute difference error loss** if the sample consists of one observation X =1? [10]

=== END OF PAPER===

TOTAL MARKS: 100